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Fractional boundary value problem in Orlicz spaces

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Abstract

The aim of this paper is devoted to investigate the existence of positive continuous solutions for boundary value problem of fractional type

$$\begin{cases} \dfrac{d^{\alpha,g}x(t)}{dt^{\alpha}} = \lambda u(t,x(t))[\zeta(x(t)) + \eta(x(t))], \quad t \in [a,b], \ 0 < \alpha < 1, \quad \lambda \in \mathbb{R}^+, \\ x(a) - px(b) = h, \end{cases} \tag{1}$$

under the monotonicity conditions imposed on η and ζ. Here $h \in \mathbb{R}^+, p \in [0,1)$, and u is "possibly singular" function from an appropriate Orlicz space. By the singularity of the above problem, we mean that the possibility of $\eta(0)$ being undefined is allowed.

To encompass the full scope of this paper, we present some examples illustrating the main results.

1 Introduction and Preliminaries

The topic fractional calculus in Orlicz spaces was introduced for the first time (if not earlier) by Richard O'Neil [16]. This topic has provoked some interest by many authors (see e.g. [15, 22, 25] and the references). As far as we know, there are only very few results taking into account the specific features of fractional calculus in Orlicz spaces.

In this pages we have tried to make the discussion as self-contained and synthetic as possible. We hope for the indulgence of the reader acquainted with A. Kilbas's books,

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in particular with [12, 20]. So, instead of spending a lot of time going over background material, we go directly into the primary subject matter.
Throughout this article, $C[a,b]$ denote the Banach space of continuous functions on $[a,b]$ endowed by the norm $\|x\| = \max_{t\in[a,b]} |x(t)|$. Also, we will let $L_p[a,b]$, $(1 \leq p < \infty)$ denotes the Banach space of of p-integrable functions on $[a,b]$ endowed standard norm $\|.\|_p$.

Definition 1. Let $x : [a,b] \to \mathbb{R}$, $[a,b]$ be a finite or infinite interval of the real line $\mathbb{R}$ $(-\infty \leq a < b \leq \infty)$. Also let $g(\cdot)$ be an increasing and positive continuous function on $[a,b]$, having a continuous derivatives $g'(\cdot) \neq 0$ on (a,b) such that $g(a) = 0$. We consider the Stieltjes-type fractional integrals, namely, the left-sided fractional integrals of a function x with respect to another function g on $[a,b]$, which is defined for instance in [3, 12, 23, 24] by

$$\mathcal{I}_a^{\alpha,g} x(t) := \frac{1}{\Gamma(\alpha)} \int_a^t \frac{x(s)}{(g(t)-g(s))^{1-\alpha}} g'(s)\, ds, \quad (-\infty \leq a < b \leq \infty), \quad \alpha > 0. \tag{2}$$

For complementness, we define $\mathcal{I}_a^{\alpha,g} x(a) := 0$.

Definition 1 allows us to unify different fractional integral for real-valued functions and consequently, by unified manner, to solve some boundary value problems with different types of fractional integrals and derivatives. Clearly, it is not only an unification, but we extend existing results too. For instance, the special case $g(t) = t,\ t \in [0,1]$ or $g(t) = \ln t,\ t \in [1,e]$ give the classical fractional integral operators: the Riemann-Liouville and the Hadamard ones.

Definition 2. [7] (The generalized Hölder spaces).
Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a Hölder function, i.e., ψ is increasing and continuous with $\psi(0) = 0$. The (generalized) Hölder space H_g^ψ consists of all $x \in C[a,b]$ satisfying

$$\|x(t) - x(s)\| \leq L\psi(|g(t) - g(s)|), \quad L > 0,$$

equipped with the norm

$$\|x\|_\psi := \max_{t\in I} \|x(t)\| + [x]_\psi,$$

where

$$[x]_\psi := \frac{\|x(t) - x(s)\|}{\psi(|g(t) - g(s)|)}$$

. The space H_g^ψ is a Banach space, called the generalized Hölder space and its elements are called generalized Hölder functions.
The particular if $g(t) = t$ and $\psi(t) = t^\alpha$, $\alpha \in (0,1]$ then, of course, we get the classical Hölder space.

Let us start with some basic information about Orlicz spaces [13, 17].
Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a Young's function, i. e. ψ is increasing, even, convex, and continuous with $\psi(0) = 0$ and $\lim\limits_{u\to\infty} \psi(u) = \infty$. The Orlicz space $L_\psi = L_\psi([a, b], \mathbb{R})$ is a Banach space consists of all (classes of) measurable functions $x : [a, b] \to \mathbb{R}$ for which there exists a number $k > 0$ such that

$$\int_a^b \psi\left(\frac{|x(t)|}{k}\right)\, ds \ \le 1.$$

The (Luxemburg) norm $\|x\|_\psi$ is defined as the inf of such k (see e.g. [13], [17] and the references therein for background on these topics). The Young complement (or Fenchel transform) $\widetilde{\psi}$ of ψ is defined for $u \in \mathbb{R}$ by

$$\widetilde{\psi}(u) := \sup_{v\ge 0}\ \{|u|v - \psi(v)\}.$$

Let us recall that, on the finite measure spaces, if ψ is not the null function then $C[a, b] \subset L_\infty[a, b] \subset L_\psi[a, b] \subset L_1[a, b]$. However, Lp spaces are special cases of Orlicz spaces (for $\psi(x) = \frac{|x|^p}{p}$).
As every body knew, the function from usual Lebesgue spaces L_p has at most polynomial growth. By using the properties of Orlicz spaces, we may relax this requirement, so our assumptions will be less restrictive than the standard ones. However, in view of
Theorem 1 and Remark 2 below, we know that the fractional integral operator maps all elements from particular Orlicz space into the space of continuous functions for any $\alpha \in (0, 1)$. This property fails in the case of Lebesgue spaces L_p: Recall, that the image of $\mathcal{I}_a^{\alpha,g}$ of L_p is in $C[a, b]$ if $p > \frac{1}{\alpha}$, for instance $\mathcal{I}_a^{\alpha,g} : L_2 \to C[a, b]$ for $\alpha \in (0.5, 1)$ (see e.g. Remark 1 below).
This seems to be a good place to put the following observation.

Proposition 1. [8] For any $t \in \mathbb{R}^+$, $\gamma \in (0, 1)$ and any Young's function ψ the set

$$N_t^\psi := \{k > 0 : \int_0^{tk} \psi(s^{-\gamma})\, ds \le k\} = \{k > 0 : k^{-1} \int_0^{tk} \psi(s^{-\gamma})\, ds \le 1\},$$

is nonempty.

Proposition 2. [8] Let $\gamma > 0$. For any Young's function ψ with Young's complement $\widetilde{\psi}$ satisfying

$$\int_0^t \widetilde{\psi}(s^{-\gamma})\, ds < \infty, \qquad t > 0, \tag{3}$$

the function $\widetilde{\Psi} : [0, \infty) \to [0, \infty)$ defined by

$$\widetilde{\Psi}_\gamma(t) := \inf\{k > 0 : \int_0^{tk^{\frac{1}{\gamma}}} \psi(s^{-\gamma})\, ds \le k^{\frac{1}{\gamma}}\}, \quad t \ge 0, \tag{4}$$

is increasing and continuous with $\widetilde{\Psi}_\gamma(0) = 0$.

Proposition 3. [8] Let $\gamma > 0$. If ψ is a Young's-type function with Young's complement $\widetilde{\psi}$ such that

$$\int_0^t \widetilde{\psi}(s^{-\gamma})\, ds < \infty, \qquad t > 0, \tag{5}$$

then

$$\int_0^t \widetilde{\psi}(s^{-\alpha})\, ds < \infty, \qquad t > 0, \qquad \text{holds for any } \alpha \in (0, \gamma]. \tag{6}$$

Remark 1. [8] We remark that: If we fix $p > 1$, it can be easily seen that (3) holds with $\psi_p(u) = \frac{|u|^p}{p}$ for all $\gamma \in (0, 1 - \frac{1}{p})$. In other words, if we fix $\gamma \in (0, 1)$, then (3) holds with $\psi_p(u) = \frac{|u|^p}{p}$ for all $p > \frac{1}{1-\gamma}$. However, if $p > \frac{1}{1-\gamma}$, $\gamma \in (0, 1)$, it is not hard to see that

$$\widetilde{\Psi}_\gamma(t) = \frac{t^{\frac{1}{q}-\gamma}}{\sqrt[q]{q[1 - q\gamma]}}, \qquad t \geq 0. \tag{7}$$

Remark 2. [8] It is also worth to remark (see e.g. [13, page 14]) that the Young's function ψ defined by $\psi(u) := e^{|u|} - |u| - 1$, has the Young's complement $\widetilde{\psi} = (1+|u|)\log(1+|u|) - |u|$ satisfies (*cf.* [8, Example 1]) (3). This is may be combined with the definition of $\widetilde{\Psi}_\gamma$ in order to assure that,

$$\widetilde{\Psi}_\gamma(1) = \inf\{k > 0 : \int_0^{k^{\frac{1}{\gamma}}} \widetilde{\psi}(s^{-\gamma})\, ds \leq k^{\frac{1}{\gamma}}\} = \frac{1}{e^{1-\gamma} - 1}, \tag{8}$$

holds for any $\gamma \in (0, 1)$: To see this, define the continuous function $\widetilde{\Theta} : (0, \infty] \to \mathbb{R}^+$ by

$$\widetilde{\Theta}(u) := \tilde{\psi}\left(u^{-\gamma}\right) - 1 = (1 + u^{-\gamma})\log(1 + u^{-\gamma}) - u^{-\gamma} - 1 = (1 + u^{-\gamma})\left[\log\left(1 + u^{-\gamma}\right) - 1\right].$$

Reasoning as in [8, Example 1], we know that, for any $t > 0$ the integral $\int_0^t \widetilde{\Theta}(u)\, du$ is finite for every $\gamma \in (0, 1)$. Also, it is clear that

$$\widetilde{\Theta}(u) < 0, \text{ for } u \in \left(0, (e-1)^{\frac{1}{-\gamma}}\right), \ \widetilde{\Theta}(u) = 0 \text{ for } u = (e-1)^{\frac{1}{-\gamma}}, \text{ and } \widetilde{\Theta}(u) > 0 \text{ for } u > (e-1)^{\frac{1}{-\gamma}},$$

and for any $t > 0$, a trivial calculation using integration by parts yields

$$\left|\int_0^t \widetilde{\Theta}(u)\ du\right| \leq \left(\frac{t^{1-\gamma}}{1-\gamma} + t\right)\left[\log\left(1 + t^{-\gamma}\right) - (1-\gamma)\right].$$

From which, it can be easily seen (in view of $(e^{1-\gamma} - 1)^{\frac{-1}{\gamma}} > (e-1)^{\frac{-1}{\gamma}}$) that

$$\int_0^{k^{\frac{1}{\gamma}}} \widetilde{\Theta}(u)\ du = 0, \quad \int_0^t \widetilde{\Theta}(u)\ du < 0 \text{ for } t > k^{\frac{1}{\gamma}}, \text{ and } \int_0^t \widetilde{\Theta}(u)\ du > 0 \text{ for } u \in \left(0, k^{\frac{1}{\gamma}}\right),$$

where $k = \frac{1}{e^{1-\gamma}-1}$.

In other words the smallest $t > 0$ for which $\int_0^t \widetilde{\Theta}(u^{-\gamma})du \leq t$ equals $\left(\frac{1}{e^{1-\gamma}-1}\right)^{\frac{1}{\gamma}}$. Hence $\widetilde{\Theta}_\gamma(1) = \frac{1}{e^{1-\gamma}-1}$ as claimed in (8).

The following result extended a similar result proved in [1]

Proposition 4. Let $\alpha > 0$ then, the $g-$fractional integral operator maps the a.e. nonnegative, nondecreasing real-valued functions into functions of the same type.

Proof. Let x be a.e. nonnegative, nondecreasing real-valued function on $[a, b]$. Since, the inverse function g^{-1} of g is increasing then, for any $t_1,\ t_2 \in [a, b]$ with $t_1 < t_2$ we have

$$\begin{aligned} \mathcal{I}_a^{\alpha,g} x(t_1) &= \frac{1}{\Gamma(\alpha)} \int_a^{t_1} \frac{x(s)}{(g(t_1) - g(s))^{1-\alpha}} g'(s)\, ds \\ &= \frac{1}{\Gamma(\alpha)} \int_0^{g(t_1)} \frac{x(g^{-1}(g(t_1) - s))}{s^{1-\alpha}}\, ds \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^{g(t_2)} \frac{x(g^{-1}(g(t_2) - s))}{s^{1-\alpha}}\, ds \\ &= \frac{1}{\Gamma(\alpha)} \int_a^{t_2} \frac{x(s)}{(g(t_2) - g(s))^{1-\alpha}} g'(s)\, ds = \mathcal{I}_a^{\alpha,g} x(t_2). \end{aligned}$$

This yields $0 \leq \mathcal{I}_a^{\alpha,g} x(t_1) \leq \mathcal{I}_a^{\alpha,g} x(t_2)$ for any a.e. nonnegative, nondecreasing function x, which is what we wished to show. □

Now, we introduce the following interesting theorem of $g-$fractional integral operator, which will be a basic tool for achieving our aims. Indeed, the following theorem dealing with the statements revealing how much the fractional integral $\mathcal{I}_a^{\alpha,g}$ is "better", in the sense of space inclusions, than the function x.

Theorem 1. *Let $\alpha \in (0, 1]$. For any Young's function ψ with Young's complement $\widetilde{\psi}$ satisfies*

$$\int_0^t \widetilde{\psi}(s^{\alpha-1})\, ds < \infty, \qquad t > 0, \tag{9}$$

the operator $\mathcal{I}_a^{\alpha,g}$ maps $L_\psi[a, b]$ into $H_g^{\widetilde{\Psi}_\alpha}[a, b]$. Here $\widetilde{\Psi}_\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ is defined as

$$\widetilde{\Psi}_\alpha(t) := \inf\{k > 0 : k^{\frac{1}{\alpha-1}} \int_0^{tk^{\frac{1}{1-\alpha}}} \psi(s^{\alpha-1})\, ds \leq 1\}, \quad t \geq 0 \tag{10}$$

Proof. At the beginning, we remark that (*cf.* [8, Proposition 2.]) the function $\widetilde{\Psi}_\alpha$ defined as in (10) is a Hölder function, i.e., $\widetilde{\Psi}_\alpha$ for any $\alpha \in (0, 1]$ is well defined, increasing and continuous with $\widetilde{\Psi}_\alpha(0) = 0$. In other words, the space $H_g^{\widetilde{\Psi}_\alpha}[a, b]$ is generalized Hölder space. Now, let $x \in L_\psi[a, b]$, $\alpha \in (0, 1)$ and define $w : [a, b] \to \mathbb{R}^+$ by

$$w(s) := \begin{cases} (g(t) - g(s))^{\alpha-1} g'(s) & s \in [a, t],\ t > a \\ 0 & \text{otherwise.} \end{cases}$$

We claim that $w \in L_{\widetilde{\psi}}[a,b]$, once our claim is established. The operator $\mathcal{I}_a^{\alpha,g}$ makes sense (i.e., $\mathcal{I}_a^{\alpha,g}x$ exists for every $x \in L_\psi[a,b]$, and observe that for any $t \in I$ the function

$$w_t(\eta) := \eta - \frac{1}{\|g'\|}\int_0^{\eta g(t)} \widetilde{\psi}(s^{\alpha-1})\,ds,$$

has a positive derivative for some sufficiently big $\eta>0$ (because $\widetilde{\psi}(u) \to 0$ as $u \to 0$). consequently, for any $t \in I$ there is a sufficiently big $\eta>0$ such that $w_t(\eta)>0$ and thus for any $t \in I$

$$\left\{k>0 : \frac{1}{\|g'\|}\int_0^{(\frac{k}{\|g'\|})^{\frac{1}{1-\alpha}} g(t)} \widetilde{\psi}(s^{\alpha-1})\,ds \le (\frac{k}{\|g'\|})^{\frac{1}{1-\alpha}}\right\} \neq \emptyset.$$

This is line with the following observations that they give

$$\begin{aligned}
\int_a^t \widetilde{\psi}\left(\frac{|w(s)|}{k}\right) ds &:= \int_a^t \widetilde{\psi}\left(\frac{|(g(t)-g(s))^{\alpha-1}|\,|g'(s)|}{k}\right) ds \\
&= \int_a^t \widetilde{\psi}\left(\frac{|(g(t)-g(s))^{\alpha-1}|\,\|g'\|}{k}\frac{|g'(s)|}{\|g'\|}\right) ds \\
&\le \int_a^t \widetilde{\psi}\left(\frac{|(g(t)-g(s))^{\alpha-1}|\,\|g'\|}{k}\right)\frac{|g'(s)|}{\|g'\|}\,ds \\
&= \frac{\left(\frac{k}{\|g'\|}\right)^{\frac{1}{\alpha-1}}}{\|g'\|}\int_0^{(\frac{k}{\|g'\|})^{\frac{1}{1-\alpha}} g(t)} \widetilde{\psi}(s^{\alpha-1})\,ds,
\end{aligned}$$

holds for any $k>0$, so $w \in L_{\widetilde{\psi}}[a,b]$.
Next, we prove the operator $\mathcal{I}_a^{\alpha,g}$ is well-defined:
Without loss of generality, let $a \le t_1 \le t_2 \le b$, there is no difficulty to write the following chain of inequalities

$$\begin{aligned}
|\mathcal{I}_a^{\alpha,g}x(t_2) - \mathcal{I}_a^{\alpha,g}x(t_1)|\Gamma(\alpha) &\le \left(\int_a^{t_1} \left|(g(t_2)-g(s))^{\alpha-1} - (g(t_1)-g(s))^{\alpha-1}\right| |g'(s)||x(s)|\,ds\right. \\
&+ \left.\int_{t_1}^{t_2} (g(t_2)-g(s))^{\alpha-1}|g'(s)||x(s)|\,ds\right) \\
&= \int_a^b [z_1(s)+z_2(s)]|x(s)|\,ds.
\end{aligned}$$

where

$$z_1(s) := \begin{cases} |(g(t_2)-g(s))^{\alpha-1} - (g(t_1)-g(s))^{\alpha-1}|\,|g'(s)| & s \in [a,t_1], \\ 0 & \text{otherwise} \end{cases}$$

and

$$z_2(s) := \begin{cases} (g(t_2) - g(s))^{\alpha-1}|g'(s)| & s \in [t_1, t_2], \\ 0 & \text{otherwise.} \end{cases}$$

We claim that $z_j \in L_{\widetilde{\psi}}([a, b])$, $\{j = 1, 2\}$. Once our claim is established, we conclude (in view of Hölder inequality in Orlicz space) that

$$|\mathcal{I}_a^{\alpha,g} x(t_2) - \mathcal{I}_a^{\alpha,g} x(t_1)| \leq \frac{2\left[\|z_1\|_{\widetilde{\psi}} + \|z_2\|_{\widetilde{\psi}}\right]}{\Gamma(\alpha)} \|x\|_{\psi}. \tag{11}$$

It remain to prove our claim by showing that $z_j \in L_{\widetilde{\psi}}([a, b])$, $\quad j = 1, 2$. To see this fix $a \leq t_1 \leq t_2 \leq b$ and $k > 0$. An appropriate substitution using the properties of Young functions leads to the following estimate

$$\begin{aligned}
\int_a^b \widetilde{\psi}\left(\frac{|z_1(s)|}{k}\right) ds &:= \int_a^{t_1} \widetilde{\psi}\left(\frac{|(g(t_2) - g(s))^{\alpha-1} - (g(t_1) - g(s))^{\alpha-1}|\,|g'(s)|}{k}\right) ds \\
&= \int_a^{t_1} \widetilde{\psi}\left(\frac{|(g(t_2) - g(s))^{\alpha-1} - (g(t_1) - g(s))^{\alpha-1}|\,\|g'\|}{k}\frac{|g'(s)|}{\|g'\|}\right) ds \\
&\leq \int_a^{t_1} \widetilde{\psi}\left(\frac{[(g(t_1) - g(s))^{\alpha-1} - (g(t_2) - g(s))^{\alpha-1}]\,\|g'\|}{k}\right)\frac{|g'(s)|}{\|g'\|} ds \\
&\leq \int_a^{t_1} \widetilde{\psi}\left(\frac{[(g(t_1) - g(s))^{\alpha-1}]\,\|g'\|}{k}\right)\frac{|g'(s)|}{\|g'\|} ds \\
&- \int_a^{t_1} \widetilde{\psi}\left(\frac{[(g(t_2) - g(s))^{\alpha-1}]\,\|g'\|}{k}\right)\frac{|g'(s)|}{\|g'\|} ds \\
&\leq \frac{\left(\frac{k}{\|g'\|}\right)^{\frac{1}{\alpha-1}}}{\|g'\|}\left[\int_0^{(\frac{k}{\|g'\|})^{\frac{1}{1-\alpha}} g(t_1)} \widetilde{\psi}(s^{\alpha-1})\, ds - \int_{(\frac{k}{\|g'\|})^{\frac{1}{1-\alpha}}(g(t_2)-g(t_1))}^{(\frac{k}{\|g'\|})^{\frac{1}{1-\alpha}} g(t_2)} \widetilde{\psi}(s^{\alpha-1})\, ds\right] \\
&= \frac{\left(\frac{k}{\|g'\|}\right)^{\frac{1}{\alpha-1}}}{\|g'\|}\left[\int_0^{(\frac{k}{\|g'\|})^{\frac{1}{1-\alpha}} g(t_1)} \widetilde{\psi}(s^{\alpha-1})\, ds - \int_0^{(\frac{k}{\|g'\|})^{\frac{1}{1-\alpha}} g(t_2)} \widetilde{\psi}(s^{\alpha-1})\, ds\right. \\
&+ \left.\int_0^{(\frac{k}{\|g'\|})^{\frac{1}{1-\alpha}}(g(t_2)-g(t_1))} \widetilde{\psi}(s^{\alpha-1})\, ds\right] \\
&\leq \frac{\left(\frac{k}{\|g'\|}\right)^{\frac{1}{\alpha-1}}}{\|g'\|}\int_0^{(\frac{k}{\|g'\|})^{\frac{1}{1-\alpha}}(g(t_2)-g(t_1))} \widetilde{\psi}(s^{\alpha-1})\, ds.
\end{aligned}$$

Now in account of (4), we can conclude that $z_1 \in L_{\widetilde{\psi}}(I)$ and the definition of norm in Orlicz space we have,

$$\begin{aligned}
\|z_1\|_{\widetilde{\psi}} &= \inf\{k>0 : \int_a^b \widetilde{\psi}\left(\frac{|z_1(s)|}{k}\right) ds \le 1\} \\
&= \|g'\| \inf\{\frac{k}{\|g'\|} > 0 : \int_a^b \widetilde{\psi}\left(\frac{|z_1(s)|}{k}\right) ds \le 1\} \\
&\le \widetilde{\Psi}_\alpha\left(|g(t_2)-g(t_1)|\right).
\end{aligned}$$

Also, with some further efforts one can get $z_2 \in L_{\widetilde{\psi}}(I)$ and

$$\|z_2\|_{\widetilde{\psi}} \le \widetilde{\Psi}_\alpha\left(|g(t_2)-g(t_1)|\right).$$

Thus, equation (11) takes the form

$$\|\mathcal{I}_a^{\alpha,g}x(t_2) - \mathcal{I}_a^{\alpha,g}x(t_1)\| \le \frac{4\widetilde{\Psi}_\alpha\left(|g(t_2)-g(t_1)|\right)}{\Gamma(\alpha)} \|x\|_\psi . \tag{12}$$

Also

$$[\mathcal{I}_a^{\alpha,g}x]_{\widetilde{\Psi}_\alpha} \le \frac{4}{\Gamma(\alpha)} \|x\|_\psi .$$

Moreover, in view of our definition $\mathcal{I}_a^{\alpha,g}x(a) := 0$, we observe that

$$\|\mathcal{I}_a^{\alpha,g}x(t)\| = \|\mathcal{I}_a^{\alpha,g}x(t) - \mathcal{I}_a^{\alpha,g}x(a)\| \le \widetilde{\Psi}_\alpha(\|g\|)[\mathcal{I}_a^{\alpha,g}x]_{\widetilde{\Psi}_\alpha}.$$

We finally get

$$\|\mathcal{I}_a^{\alpha,g}x\|_{\widetilde{\Psi}_\alpha} \le \frac{4}{\Gamma(\alpha)} \|x\|_\psi \left(1+\widetilde{\Psi}_\alpha(\|g\|)\right).$$

In view of (12), the continuity of $\widetilde{\Psi}_\alpha$ and the definition $\mathcal{I}_a^{\alpha,g}x(a) := 0$ lead to the norm continuity of $\mathcal{I}_a^{\alpha,g}x$. Thus, the map $\mathcal{I}_a^{\alpha,g} : L_\psi[a,b] \to H_g^{\widetilde{\Psi}_\alpha}[a,b]$ is well-defined and the theorem is then proved.

□

From now, the definitions of the $g-$fractional derivatives of x becomes a natural requirement.

Definition 3. Let $x : [a,b] \to \mathbb{R}$ and g be an increasing and positive function on $(a,b]$, having a continuous derivative g' on (a,b). The $g-$fractional derivative of a given function x of order $\alpha \in (n-1,n)$ is define by

$$\mathcal{D}^{\alpha,g}x(t) := \left(\frac{1}{g'(t)}D\right)^n \mathcal{I}_a^{n-\alpha,g}x(t). \tag{13}$$

Definition 4. Let $x : [a,b] \to \mathbb{R}$ and g be an increasing and positive function on $(a,b]$, having a continuous derivative g' on (a,b). The $g-$Caputo fractional derivative of a given function x of order α is define by

$$\frac{d^{\alpha,g}}{dt^\alpha}x(t) := \mathcal{I}_a^{n-\alpha,g}\left(\frac{1}{g'(t)}D\right)^n x(t). \tag{14}$$

Remark 3. This seems to be a good place to remark that, conditions needed for the existence of $g-$Caputo fractional derivative are very restrictive. A very rough condition which ensures the existence of $\frac{d^{\alpha,g}}{dt^{\alpha,g}}x$ is that $x \in AC^{n-1}[a,b]$. In other words, the $g-$Caputo-type fractional derivative has the disadvantage that it completely loses its meaning if $D^{n-1}x$ fails to be (almost everywhere) differentiable on $[a,b]$. We also remark that, in the space of continuous functions, the $g-$type Riemann-Liouville fractional differential operator is left inverse of the corresponding $g-$ fractional integral operator. For this reason, in the space of continuous function, the fractional-type integral equations involving the $g-$ fractional integral operators and the corresponding $g-$type Riemann-Liouville fractional differential operator are equivalent. Unfortunately, the $g-$Caputo-type fractional differential operator does not enjoy the same "nice" behavior. Indeed, even for the Hölder functions, outside of the space of absolutely continuous functions, the $g-$Caputo-type fractional differential operator is not (in general) left inverse to the corresponding $g-$ fractional integral operator. In other wards, even in Hölder spaces, the equivalence between the fractional-type integral equations involving the $g-$ fractional integral operators and the corresponding $g-$Caputo-type fractional differential problem can be lost. This goes back to the old-known fact that fractional integral operator is a continuous mapping from Hölder spaces "onto" Hölder spaces (which, of course, contains also continuous nowhere differentiable functions), see e.g. [20, Theorem 13.13]. Indeed, in what follows, we will show that even in the context of real-valued Hölder functions the converse implication from the fractional integral equations to the corresponding Caputo-type differential form is no longer necessarily true. To see this, we consider the particular form of the fractional integral operator when $g(t) = t, \quad t \in [0,1]$) and $\alpha \in (0,1)$: Let x be Hölder function (but nowhere differentiable on $[0,1]$) function of some critical order $\gamma < 1$. According to [20, Theorem 13.13] we know that, there is $\alpha \in (0,1)$ depends only on γ and a Hölder function $y \notin AC[0,1]$ such that $\mathcal{I}_0^{\alpha,t}y = x$. Form which we conclude that $\frac{d^{\alpha,t}}{dt^{\alpha}}\mathcal{I}_0^{\alpha,t}y = \frac{d^{\alpha,t}}{dt^{\alpha}}x$ is "meaningless". This gives a reason to believe that, even on the Hölder spaces (but out of the space of absolutely continuous functions), $\frac{d^{\alpha}}{dt^{\alpha}}$ is not left inverse of $\mathcal{I}_0^{\alpha,t}y$ as required.

In this connection, it can be easily seen that some of the papers (such as [2, 5, 6, 9, 11, 14, 19, 18, 21, 26, 27] in case of real-valued functions) contain an error in the proof of the equivalent of the fractional-type differential problems and the corresponding integral forms. Indeed, we will show below that the existence of continuous solutions of the fractional-type integral problems is not sufficient to ensure the existence of solutions to the corresponding the Caputo fractional differential problems. Here, we simply overcome such an equivalence problem by applying a know connection between the Riemann-Liouville and Caputo fractional differential operators (see for instance [3, Theorem 3.]) i.e., we put

$$\frac{d^{\alpha,g}}{dt^{\alpha}}x(t) := \mathcal{D}^{\alpha,g}x(t) - \sum_{k=0}^{n-1}\frac{(g(t))^{k-\alpha}}{\Gamma(1+k-\alpha)}((\frac{1}{g'(t)}D)^k x)(a). \tag{15}$$

Here $\mathcal{D}^{\alpha,g}$ stands the $g-$fractional differential operator. Reasoning as in [3, Theorem 3.], we know that, for the function $x \in AC^n[a,b]$, the definition (15) coincides with the usual (Definition 4) of the $g-$Caputo fractional derivative.

Theorem 2. (Schauder fixed point theorem)[10]
Let U be a convex subset of a Banach space F, and $T : U \to U$ is compact, continuous map. Then T has at least one fixed point in U.

2 Positive, Continuous-Solution to Caputo BVP (1)

In this section, we restrict our attention to discuss the existence of positive, continuous solution to the fractional boundary value problem combined with the boundary condition (1). To do this, we make use of the Schauder's fixed point theorem.
Now, we proceed to obtain (formally) the integral equation (modeled off the problem (1)). keeping in mind the continuity of $\mathcal{I}_a^{\alpha,g}u(\cdot,x(\cdot))\left[\xi(x(\cdot))+\eta(x(\cdot))\right]$, we have

$$x(t) = x(a) + \lambda\mathcal{I}_a^{\alpha,g}u(t,x(t))\left[\xi(x(t))+\eta(x(t))\right], \tag{16}$$

with some (presently unknown) quantity $x(a)$. At $t = b$, this reads

$$x(b) = x(a) + \lambda\mathcal{I}_a^{\alpha,g}u(b,x(b))\left[\xi(x(b))+\eta(x(b))\right].$$

We can plug this into the boundary condition $x(a) - px(b) = h$ and derive

$$x(a) = \frac{h}{1-p} + \frac{\lambda p}{1-p}\mathcal{I}_a^{\alpha,g}u(b,x(b))\left[\xi(x(b))+\eta(x(b))\right]. \tag{17}$$

Now inserting $x(a)$ into (16) yields the integral equation

$$x(t) = \frac{h}{1-p} + \lambda\mathcal{I}_a^{\alpha,g}u(t,x(t))\left[\xi(x(t))+\eta(x(t))\right] + \frac{\lambda p}{1-p}\mathcal{I}_a^{\alpha,g}u(b,x(b))\left[\xi(x(b))+\eta(x(b))\right], \quad t\in[a,b], \alpha\in(0,1). \tag{18}$$

with $\xi : [0,\infty] \to [0,\infty]$ is continuous non-decreasing and $\eta : (0,\infty] \to [0,\infty]$ is continuous non-increasing and $u : [a,b]\times\mathbb{R} \to \mathbb{R}$ belongs to an appropriate Orlicz space.

The preceding investigation would lead to a local existence theorem for the *possibly singular* fractional differential equation

$$\frac{d^{\alpha,g}x(t)}{dt^{\alpha}} = \lambda u(t,x(t))\left[\xi(x(t))+\eta(x(t))\right], \quad \alpha\in(0,1), \quad t\in[a,b], \quad \lambda\in\mathbb{R}^+, \tag{19}$$

combined with appropriate boundary condition

$$x(a) - px(b) = h, \tag{20}$$

with certain constants $h \in \mathbb{R}^+$, $p \in [0,1)$. Here $\frac{d^{\alpha,g}}{dt^{\alpha}}$ stands the classical Caputo- Fractional differential operator.

By the singularity of the problems (18) and (1), we mean that the possibility of $\eta(0)$ being undefined is permitted. The possibility that the function u is Carathéodory-type singular with respect to t is allowed as well: Indeed, the real advantage of the Carathéodory functions is that it can be singular with respect to t.

To facilitate our discussion, let ψ be a Young's function with Young's complement $\widetilde{\psi}$ satisfies

$$\int_0^t \widetilde{\psi}(s^{\alpha-1})\, ds < \infty, \qquad t > 0.$$

Suppose $u : [a, b] \times \mathbb{R} \to \mathbb{R}$ be given function such that

1. u is a Carathéodory function on $[a, b] \times \mathbb{R}$ (that is, for any $x \in \mathbb{R}$ the function $u(\cdot, x)$ is measurable on $[a, b]$ and for almost every $t \in [a, b]$, $u(t, \cdot)$ is continuous on $\mathbb{R}$) such that $u : [a, b] \times \mathbb{R}^+ \to \mathbb{R}^+$.
2. For any $r > 0$ there exists $M_r \in L_\psi([a, b], (0, \infty))$ such that $|u(t, x)| \leq M_r(t), \quad t \in [a, b]$ and $|x| \leq r$.

Let us state and prove the following result [4].

Lemma 1. *If $u : [a, b] \times \mathbb{R} \to \mathbb{R}$ satisfy the assumptions (1)-(2). Then*

1. $M_r(t) \geq \max_{|x| \leq r} |u(t, x)|, \quad r > 0,\ t \in [a, b],$
2. *For each* $x \in C[a, b],\ u(\cdot, x(\cdot)) \in L_\psi([a, b], \mathbb{R}),$
3. $\|u(\cdot, x(\cdot))\|_\psi \leq \|M_r\|_\psi$, *for any* $x \in Q_r := \{x \in C[a, b] : \|x\| \leq r,\ r > 0\}$.

Proof. First, we observe that, the separability of $[a, b]$ gives a reason to believe that M_r is measurable. Fix $r > 0$, let $x \in Q_r$ and note that

$$\int_a^b \psi\left(\frac{u(t, x(t))}{\|M_r\|_\psi}\right) dt \leq \int_a^b \psi\left(\frac{M_r(t))}{\|M_r\|_\psi}\right) dt \leq 1.$$

This yields $u(\cdot, x(\cdot)) \in L_\psi$ and $\|u\|_\psi \leq \|M_r\|_\psi$. □

As an example of a natural mapping satisfies assumptions (1)-(2), we have the following example:

Example 2.1. Define the Carathéodory function

$$u(t, x) := e^{|x|} \log\left(\frac{b-a}{t-a}\right),\ t \in [a, b], \quad x \in \mathbb{R}.$$

It seems good place to remark that, the real advantage of the Carathéodory function $u \in L_\psi$ is that it is practically always satisfied: u can even grow arbitrarily fast with respect to x (e.g. exponentially) and even be singular with respect to t (depending of ψ).

In this connection, we conclude that, for any $r > 0$ and $t \in [a, b]$, we have

$$M_r(t) = e^r \log\left(\frac{b-a}{t-a}\right).$$

Now, if we choose $\psi(u) = e^{|u|} - |u| - 1$, it follows in view of

$$\int_a^b \psi\left(\frac{\log(\frac{b-a}{t-a})}{2}\right) dt = (b-a)\int_0^1 \psi\left(\frac{\log(\frac{1}{t})}{2}\right) dt = (b-a)\int_0^1 \left(\frac{1}{\sqrt{t}} - \frac{1}{2}\log\frac{1}{t} - 1\right) dt = \frac{(b-a)}{2} < 1,$$

that M_r belongs to $L_\psi[a,b]$ and $\|M_r\|_\psi \leq 2e^r$.

Now, we are in the position to state and prove the following existence theorem

Theorem 3. *Let $\alpha \in (0,1)$, $\lambda \in \mathbb{R}^+$. Let ψ be a Young's function with Young's complement $\widetilde{\psi}$ satisfies (3). Assume that the assumptions of Lemma 1 hold along with*

1. *$\xi : [0,\infty] \to [0,\infty]$ is continuous and non-decreasing,*
2. *$\eta : (0,\infty] \to [0,\infty]$ is continuous and non-increasing.*

If there exist $\gamma > h$ such that

$$h + \frac{4\lambda(\xi(\gamma) + \eta(h))\widetilde{\Psi}(\|g\|)}{\Gamma(\alpha)} \|M_\gamma\|_\psi \leq \gamma(1-p), \tag{21}$$

then, the integral equation (18) has at least one positive continuous solution x such that $0 < h \leq x(t) \leq \gamma$, for $t \in [a,b]$.

Proof. Our goal is to show that the problem (18) admits at least one fixed point. To do this, we proceed by making use of the Schauder's fixed point theorem. In order to do this, let us define the closed convex set Q (required by Schauder's fixed theorem) by:

$$Q \ := \ \{x \in C[a,b] : h \leq x(t) \leq \gamma, \ \ \forall t \in [a,b]\}.$$

Also, we define the operator $T : Q \to Q$ by

$$Tx(t) = \frac{h}{1-p} + \lambda \mathcal{I}_a^{\alpha,g} u(t,x(t))\left[\xi(x(t)) + \eta(x(t))\right] + \frac{\lambda p}{1-p}\mathcal{I}_a^{\alpha,g} u(b,x(b))\left[\xi(x(b)) + \eta(x(b))\right], t \in [a,b], \alpha \in (0,1).$$

Obviously, our assumptions imposed on ξ and η along with the fact that $u(\cdot, x(\cdot)) \in L_\psi[a,b] \subseteq L_1[a,b]$ for every $x \in C[a.b]$ give a reason ([12, Lemma 2.1.]) to believe that the operator T makes sense.

We need now to divide the proof into a few steps. In fact, we will prove the following claims:

(step 1): For any $x \in Q$, Tx is continuous on $[a,b]$,

(step 2): T leaves Q invariant (that is, $T : Q \longrightarrow Q$ is well-defined),

(step 3): $T : Q \longrightarrow Q$ is continuous,

(step 4): $T : Q \longrightarrow Q$ is compact.

To prove the assertion of (*step*1), we let $x \in Q$ and $a \le \tau \le t \le b$. It can easily seen that

$$\begin{aligned} |Tx(t) - Tx(\tau)| &= \frac{1}{\Gamma(\alpha)}\left|\int_a^t (g(t)-g(s))^{\alpha-1} g'(s) u(s,x(s))\,[\xi(x(s))+\eta(x(s))]\;ds\right. \\ &- \left.\int_a^\tau (g(\tau)-g(s))^{\alpha-1} g'(s) u(s,x(s))\,[\xi(x(s))+\eta(x(s))]\;ds\right| \\ &\le \frac{\lambda(\xi(\gamma)+\eta(h))}{\Gamma(\alpha)}\left(\int_a^\tau \left|(g(t)-g(s))^{\alpha-1} - (g(\tau)-g(s))^{\alpha-1}\right| |g'(s)||u(s,x(s))|\,ds\right. \\ &+ \left.\int_\tau^t (g(t)-g(s))^{\alpha-1}|g'(s)||u(s,x(s))|\,ds\right) \\ &= \frac{\lambda(\xi(\gamma)+\eta(h))}{\Gamma(\alpha)}\int_a^b [z_1(s)+z_2(s)]||u(s,x(s))||\,ds \\ &\le \frac{\lambda(\xi(\gamma)+\eta(\mu))}{\Gamma(\alpha)}\int_a^b [z_1(s)+z_2(s)] M_\gamma(s)\,ds. \end{aligned}$$

where

$$z_1(s) := \begin{cases} \left|(g(t)-g(s))^{\alpha-1} - (g(\tau)-g(s))^{\alpha-1}\right| |g'(s)|, & s \in [a,\tau], \\ 0, & \text{otherwise.} \end{cases}$$

and

$$z_2(s) := \begin{cases} (g(t)-g(s))^{\alpha-1}|g'(s)|, & s \in [\tau,t]. \\ 0, & \text{otherwise.} \end{cases}$$

Arguing similarly as in the proof of Theorem 1, it is not hard to see that

$$|Tx(t) - Tx(\tau)| \le \frac{4\lambda(\xi(\gamma)+\eta(h))\widetilde{\Psi}(|g(t)-g(\tau)|)}{\Gamma(\alpha)} \|M_\gamma\|_\psi\,. \tag{22}$$

From which we assure the continuity of $Tx(\cdot)$ on $[a,b]$ as needed for the assertion of (*step*1).

In this connection, in view of our definition that $\mathcal{I}_a^{\alpha,g} x(a) := 0$ for $x \in L_1[a,b]$, by using Assumption (2) of Lemma 1 it follows

$$\|Tx\| \le \frac{h}{1-p} + \frac{4\lambda}{1-p}\frac{(\xi(\gamma)+\eta(h))\widetilde{\Psi}(\|g\|)}{\Gamma(\alpha)} \|M_\gamma\|_\psi \le \gamma, \tag{23}$$

$$Tx(t) \ge \frac{h}{1-p} \ge h. \tag{24}$$

Altogether yield T maps Q into Q as needed for the assertion of (*step*2).

To prove the assertion of (*step*3), it is sufficient to choose $x_n \to x$ in Q (that is, $x_n(t) \to x(t)$ uniformly in $\mathbb{R}$). In this case, we write the following chain of inequalities

$$
\begin{aligned}
|Tx_n(t)-Tx(t)| &= \frac{\lambda}{\Gamma(\alpha)}\left|\int_a^t (g(t)-g(s))^{\alpha-1}g'(s)u(s,x_n(s))[\xi(x_n(s))+\eta(x_n(s))]\,ds\right.\\
&- \left.\int_a^t (g(t)-g(s))^{\alpha-1}g'(s)u(s,x(s))[\xi(x(s))+\eta(x(s))]\,ds\right|\\
&+ \frac{\lambda p}{(1-p)\Gamma(\alpha)}\left|\int_a^b (g(b)-g(s))^{\alpha-1}g'(s)u(s,x_n(s))[\xi(x_n(s))+\eta(x_n(s))]\,ds\right.\\
&- \left.\int_a^b (g(b)-g(s))^{\alpha-1}g'(s)u(s,x(s))[\xi(x(s))+\eta(x(s))]\,ds\right|\\
&\leq \frac{\lambda}{\Gamma(\alpha)}\int_a^t (g(t)-g(s))^{\alpha-1}|g'(s)|(|\xi(x_n(s))-\xi(x(s))|+|\eta(x_n(s))-\eta(x(s))|)\\
&\times |u(s,x_n(s))|\,ds\\
&+ \frac{\lambda}{\Gamma(\alpha)}\int_a^t (g(t)-g(s))^{\alpha-1}|g'(s)|(\xi(x(s))+\eta(x(s))]\,|u(s,x_n(s)))-u(s,x(s))|\;ds\\
&+ \frac{\lambda p}{(1-p)\Gamma(\alpha)}\int_a^b (g(b)-g(s))^{\alpha-1}|g'(s)|(|\xi(x_n(s))-\xi(x(s))|+|\eta(x_n(s))-\eta(x(s))|)\\
&\times |u(s,x_n(s))|\,ds\\
&+ \frac{\lambda p}{(1-p)\Gamma(\alpha)}\int_a^b (g(b)-g(s))^{\alpha-1}|g'(s)|(\xi(x(s))+\eta(x(s))]\,|u(s,x_n(s)))-u(s,x(s))|\;ds\\
&\leq \frac{\lambda}{\Gamma(\alpha)}\int_a^t (g(t)-g(s))^{\alpha-1}|g'(s)|[|\xi(x_n(s))-\xi(x(s))|+|\eta(x_n(s))-\eta(x(s))|)M_\gamma(s)\,ds\\
&+ \frac{\lambda(\xi(\gamma)+\eta(h))}{\Gamma(\alpha)}\int_a^t (g(t)-g(s))^{\alpha-1}|g'(s)|\,|u(s,x_n(s)))-u(s,x(s))|\;ds\\
&+ \frac{\lambda p}{(1-p)\Gamma(\alpha)}\int_a^b (g(b)-g(s))^{\alpha-1}|g'(s)|[|\xi(x_n(s))-\xi(x(s))|+|\eta(x_n(s))-\eta(x(s))|)\\
&\times M_\gamma(s)\,ds\\
&+ \frac{\lambda p(\xi(\gamma)+\eta(h))}{(1-p)\Gamma(\alpha)}\int_a^b (g(b)-g(s))^{\alpha-1}|g'(s)|\,|u(s,x_n(s)))-u(s,x(s))|\;ds.
\end{aligned}
$$

That is

$$\begin{aligned}
|Tx_n(t) - Tx(t)| &\leq [\|\xi(x_n) - \xi(x)\| + \|\eta(x_n) - \eta(x)\|] \frac{2\lambda\widetilde{\Psi}(\|g\|)}{\Gamma(\alpha)} \|M_\gamma(\cdot)\|_\psi \\
&+ \frac{\lambda(\xi(\gamma) + \eta(h))}{\Gamma(\alpha)} \int_a^t (g(t) - g(s))^{\alpha-1} |g'(s)| \, |u(s, x_n(s)) - u(s, x(s)))| \, ds \\
&+ [\|\xi(x_n) - \xi(x)\| + \|\eta(x_n) - \eta(x)\|] \frac{2\lambda p\widetilde{\Psi}(\|g\|)}{(1-p)\Gamma(\alpha)} \|M_\gamma(\cdot)\|_\psi \\
&+ \frac{\lambda p(\xi(\gamma) + \eta(h))}{(1-p)\Gamma(\alpha)} \int_a^b (g(b) - g(s))^{\alpha-1} |g'(s)| \, |u(s, x_n(s)) - u(s, x(s)))| \, ds \\
&= [\|\xi(x_n) - \xi(x)\| + \|\eta(x_n) - \eta(x)\|] \frac{2\lambda\widetilde{\Psi}(\|g\|)}{(1-p)\Gamma(\alpha)} \|M_\gamma(\cdot)\|_\psi \\
&+ \frac{\lambda(\xi(\gamma) + \eta(h))}{\Gamma(\alpha)} \int_a^t (g(t) - g(s))^{\alpha-1} |g'(s)| \, |u(s, x_n(s)) - u(s, x(s)))| \, ds \\
&+ \frac{\lambda p(\xi(\gamma) + \eta(h))}{(1-p)\Gamma(\alpha)} \int_a^b (g(b) - g(s))^{\alpha-1} |g'(s)| \, |u(s, x_n(s)) - u(s, x(s)))| \, ds,
\end{aligned}$$

which in the view of the continuity of ξ, η and $u(t, \cdot)$, $t \in [a, b]$ yields $T : Q \to Q$ is continuous.

Finally, we claim that $T : Q \longrightarrow Q$ is compact (as required for the assertion of $(step4)$). Once our claim is established, Schauder's fixed point theorem guarantees the existence of a fixed point of T (hence a continuous solution to (18)). Thus all we need to show is that $T(M)$, (where $M \subset Q$ is bounded) is a relatively compact set. A necessary and sufficient condition for this to hold is contained in the Arzelà-Ascoli theorem: We need to show that $T(M)$ is uniformly bounded and equicontinuous. But the uniform boundedness of $T(M)$ is a trivial consequence of (23) and the equicontinuity follows immediately from (22). Now we are able to apply Arzelà-Ascoli theorem to show that T is compact. Hence by Schauder's fixed point theorem, $T : Q \to Q$ has a fixed point in Q. Consequently, the integral equation (18) has a positive continuous solution. This completes the proof. □

Remarks.

1. For any $M \in L_\psi[a, b]$, a similar argument as in the proof of (22), would lead to

$$|\mathcal{I}_a^{\alpha,g} M(t) - \mathcal{I}_a^{\alpha,g} M(\tau)| \leq \frac{4\widetilde{\Psi}(|g(t) - g(\tau)|)}{\Gamma(\alpha)} \|M(\cdot)\|_\psi , \tag{25}$$

holds for any $t, \tau \in [a, b]$. This is, in view of $\mathcal{I}_a^{\alpha,g} x(a) = 0, \quad x \in L_1[a, b]$, points out a useful characterization of the fractional integral for the functions from Orlicz space: It is a long-known fact that the fractional integral enjoys the smooth property of being continuous map from $L_q[a, b]$ (where $0<a<b<\infty$) into $C[a, b]$ for some $p \in [1, \infty]$ satisfying $q>\frac{1}{\alpha+(\frac{1}{p})}$. However, the inequality (25) tells us that the FI has a similar "smoothing property" from Orlicz spaces into the space $C[a, b]$. Precisely, in view of Remark 2, it can be easily seen the following observation: There is a Young function ψ (hence an Orlicz space $L_\psi[a, b]$) such that $\mathcal{I}_a^{\alpha,g}$ maps *all* elements from $L_\psi[a, b]$ into the $C[a, b]$ for *any* $\alpha \in (0, 1)$. This property fails in the case of Lebesgue spaces L_p (see Remark 1).

2. Observe the inequality (21) and note that, for sufficiently small $\lambda > 0$, (21) solvable provided that γ is sufficiently large in a sense to be made precise. In account of this observation, we can see that the Assumption (21) is not too restrictive.

Next, we will show that every positive continuous solution of (18) is necessary solves the problem (1): Let $x \in C[a,b]$ be a solution of (18). Since x is continuous (but it is not necessary absolutely continuous) on $[a,b]$ we have to invoke the Definition (15) of the Caputo fractional derivatives as follows

$$\begin{aligned} \frac{d^{\alpha,g}x(t)}{dt^{\alpha}} &= \mathcal{D}^{\alpha,g}x(t) - \frac{g(t)^{-\alpha}}{\Gamma(1-\alpha)}x(a) \\ &= \mathcal{D}^{\alpha,g}\left[x(a) + \lambda \mathcal{I}_a^{\alpha,g}u(t,x(t))\left[\xi(x(t)) + \eta(x(t))\right]\right] - \frac{g(t)^{-\alpha}}{\Gamma(1-\alpha)}x(a) \\ &= x(a)\frac{g(t)^{-\alpha}}{\Gamma(1-\alpha)} + \lambda \mathcal{D}^{\alpha,g}\mathcal{I}_a^{\alpha,g}u(t,x(t))\left[\xi(x(t)) + \eta(x(t))\right] - \frac{g(t)^{-\alpha}}{\Gamma(1-\alpha)}x(a) \\ &= \lambda u(t,x(t))\left[\xi(x(t)) + \eta(x(t))\right]. \end{aligned} \tag{26}$$

On the other hand, equation (18) implies

$$\begin{aligned} x(a) - px(b) &= x(a) - p\left[c + \lambda \mathcal{I}_a^{\alpha,g}u(b,x(b))\left[\xi(x(b)) + \eta(x(b))\right]\right] \\ &= x(a)(1-p) - p\lambda \mathcal{I}_a^{\alpha,g}u(b,x(b))\left[\xi(x(b)) + \eta(x(b))\right]. \end{aligned} \tag{27}$$

Thus if we plug (17) into (27), we arrive at the boundary condition $x(a) - px(b) = h$. This is may be combined with (26) in order to assure the existence of positive solution $x \in C[a,b]$ to the problem (1). This completes the proof.

3 Examples

In order to encompass the full scope of this paper, we give some examples illustrating the results of Theorem 3. We start with the following (simple) example

Example 3.1. Let us consider the following *singular* problem

$$\begin{cases} \dfrac{d^{\frac{3}{4}}x(t)}{dt^{\frac{3}{4}}} = \dfrac{\log(1+x^2(t))}{10\sqrt[4]{t}}\left[\sqrt[4]{x(t)} + \dfrac{1}{10\sqrt[4]{x(t)}}\right], \quad t \in (0,1), \\ x(0) - \dfrac{1}{2}x(1) = \dfrac{1}{10}. \end{cases} \tag{28}$$

Observe that the problem (28) is a special case of (1) if we put $\alpha = \frac{3}{4}$, $g(t) = t$, $a = 0$, $b = 1$, $p = \frac{1}{2}$, $h = \lambda = \frac{1}{10}$ and

- $u(t,x) = \dfrac{\log(1+x^2)}{\sqrt[4]{t}}$ (Hence $M_r(t) = \frac{\log(1+r^2)}{\sqrt[4]{t}}$).
- $\xi(x) = \sqrt[4]{x}, \quad \eta(x) = \dfrac{1}{10\sqrt[4]{x}}$.

In what follows, we show that the functions involved in (28) satisfy the inequality (21) of Theorem 3. Firstly, we note that $M_r \in L_{\psi_2}[0,1], r > 0$, where $\psi_2(v) = \frac{|v|^2}{2}$ (hence $\widetilde{\psi_2}(v) = \psi_2(v) = \frac{|v|^2}{2}$) and $\|M_r\|_{\psi_2} = \log(1+r^2)$. Evidently, for any $k > 0$ we have

$$\int_0^1 \psi_2\left(\frac{M_r(t)}{k}\right) dt = \frac{\log^2(1+r^2)}{k^2} \Rightarrow \|M_r\|_{\psi_2} = \log(1+r^2).$$

Also in virtue of Remark 1, there is no difficulty to see that $\widetilde{\Psi}_{\frac{3}{4}}(1) = \Psi_{\frac{3}{4}}(1) = 1$. This yields

$$h + [\xi(\gamma) + \eta(h)]\frac{4\lambda\widetilde{\Psi}_{\frac{3}{4}}(1)}{\Gamma(\alpha)}\|M_\gamma\|_\psi \approx \frac{1}{10} + \frac{3.3\log(1+\gamma^2)}{10}\left[\sqrt[4]{\gamma} + \frac{\sqrt[4]{10}}{10}\right] \leq \frac{\gamma}{2},$$

holds for e.g $\gamma = 2$. Therefore, the hypotheses of Theorem 3 are satisfied, hence we conclude that the problem (28) has at least one solution $x \in C[0,1]$ such that $1/10 \leq x(t) \leq 2, \quad t \in [0,1]$.

Example 3.2. Let us consider the following *singular* problem

$$\begin{cases} \dfrac{d^{\frac{3}{4}}x(t)}{dt^{\frac{3}{4}}} = \dfrac{e^{x(t)/4}}{20\sqrt[4]{(t-1)x(t)}}, & t \in (1,e), \\ x(1) - \dfrac{1}{2}x(e) = 1. \end{cases} \tag{29}$$

Observe that the problem (29) is a special case of (1) if we put $\alpha = \frac{3}{4}$, $g(t) = \log t$, $a = 1$, $b = e$, $p = \frac{1}{2}$, $h = 1$, $\lambda = \frac{1}{20}$ and

- $u(t,x) = \dfrac{e^{x/4}}{\sqrt[4]{t-1}}$ (Hence $M_r(t) = \frac{e^{r/4}}{\sqrt[4]{t-1}}$).
- $\xi(x) = 0, \quad \eta(x) = \dfrac{1}{\sqrt[4]{x}}$.

In what follows, we show that the functions involved in (29) satisfy the inequality (21) of Theorem 3. Firstly, we note that $M_r \in L_{\psi_2}[1,e], r > 0$, where $\psi_2(v) = \frac{|v|^2}{2}$ (hence $\widetilde{\psi_2}(v) = \psi_2(v) = \frac{|v|^2}{2}$) and $\|M_r\|_{\psi_2} = \log(1+r^2)\sqrt[4]{e-1}$. Evidently, for any $k > 0$ we have

$$\int_1^e \psi_2\left(\frac{M_r(t)}{k}\right) dt = \frac{\log^2(1+r^2)\sqrt{e-1}}{k^2} \Rightarrow \|M_r\|_{\psi_2} = \log(1+r^2)\sqrt[4]{e-1}.$$

Also in virtue of Remark 1, there is no difficulty to see that $\widetilde{\Psi}_2(1) = \Psi_2(1) = 1$. This yields

$$h + [\xi(\gamma) + \eta(h)]\frac{4\lambda\widetilde{\Psi}_{\frac{3}{4}}(1)}{\Gamma(\alpha)}\|M_\gamma\|_\psi \approx 1 + \frac{3.7e^{\gamma/4}}{20} \leq \frac{\gamma}{2},$$

holds for e.g $\gamma = 4$. Therefore, the hypotheses of Theorem 3 are satisfied, hence we conclude that the problem (29) has at least one solution $x \in C[1,e]$ such that $1 \leq x(t) \leq 4, \quad t \in [1,e]$.

Example 3.3. Let us consider the following *singular* problem

$$\begin{cases} \dfrac{d^{1/2}x(t)}{dt^{1/2}} = \dfrac{e^{\frac{x(t)}{8}}}{26}\log\left(\dfrac{1}{t}\right)\left[\log(\dfrac{1}{8}+\sqrt[4]{x(t)})+\dfrac{1}{5e^{32x}\sqrt[4]{x(t)}}\right], \quad t\in(0,1), \\ x(0)-\dfrac{1}{8}x(1)=\dfrac{1}{16}. \end{cases} \tag{30}$$

Observe that the problem (30) is a special case of (1) if we put $\alpha = \frac{1}{2}$, $g(t) = t$, $a = 0$, $b = 1$, $p = \frac{1}{8}$, $h = \frac{1}{16}$, $\lambda = \frac{1}{26}$ and

- $u(t,x) = e^{x(t)/8}\log\left(\dfrac{1}{t}\right)$ (Hence $M_r(t) = e^{r/8}\log\left(\frac{1}{t}\right)$).
- $\xi(x) = \log(\dfrac{1}{8}+\sqrt[4]{x})$, $\quad \eta(x) = \dfrac{1}{5e^{32x}\sqrt[4]{x}}$.

By Example 2.1, $M_r \in L_\psi[0,1]$ and $\|M_r\|_\psi \le 2e^{r/8}$ (where $\psi(v) = e^{|v|} - |v| - 1$). Owing to (8), we have $\widetilde{\Psi}_{\frac{1}{2}}(1) = \frac{1}{\sqrt{e}-1}$ holds for $\alpha = \frac{1}{2}$.

Thus

$$h + [\xi(\gamma)+\eta(h)]\frac{4\lambda\widetilde{\Psi}_{\frac{1}{2}}(1)}{\Gamma(\alpha)}\|M_\gamma\|_\psi \approx \frac{1}{16}+\frac{1}{26}\left[\frac{4(1.5415)(2)e^{\gamma/8}}{\mathit{\Gamma}(0.5)}\right]\left[\log(\frac{1}{8}+\sqrt[4]{\gamma})+\frac{\sqrt[4]{16}}{5e^2}\right] \le \frac{7\gamma}{8}$$

holds for e.g $\gamma = 4$. Therefore, the hypotheses of Theorem 3 are satisfied, hence we conclude that the problem (30) has at least one solution $x \in C[0,1]$ such that $\frac{1}{16} \le x(t) \le 4, \quad t \in [0,1]$. And we are finished.

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